

Electromagnetism

Electromagnetism is a branch of physics which involves the study of the **electromagnetic force**, a type of physical interaction that occurs between electrically charged particles. The electromagnetic force usually exhibits electromagnetic fields, such as electric fields, magnetic fields, and light.

Ampere's Circuital Law

Ampere's law states that the line integral of the magnetic field \vec{B} around any **closed circuit** is equal to μ_0 (permeability constant) times the total current 'I' threading or passing through this closed circuit. The closed circuit is called the **Amperean loop**. Mathematically,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \quad \dots (1)$$

In classical electromagnetism, **Ampere's circuital law** relates the integrated magnetic field around a closed loop to the electric current passing through the loop. **James Clerk Maxwell** (not Ampere) derived it using hydrodynamics in 1861 and it is now one of the Maxwell equations, which form the basis of classical electromagnetism.

Ampere's law finds its applications in deriving the magnetic field intensity of certain loops, solenoids, straight conductors, etc.

Inconsistency of Ampere's Circuital Law:

According to Ampere's Circuital Law, the line integral of the magnetic field \vec{B} along any closed loop C is proportional to current 'I' passing through the closed loop.

In 1864, Maxwell showed that the above equation (1) is logically **inconsistent**. To prove this inconsistency, we consider a parallel plate capacitor being charged by a battery as shown. As the charging continues, a current I flows through the connecting wires, which of course changes with time. This current produces a magnetic field around the capacitor. Consider two planar loops C_1 and C_2 ; C_1 just left of the capacitor and C_2 in between the capacitor plates, with their planes parallel to these plates.

Now the current I flows across the area bounded by loop C_1 because connecting wire passes through it. Hence from Ampere's law, we have,

$$\oint_{C_1} \vec{B} \cdot d\vec{l} = \mu_0 I \quad \dots (2)$$

But the area bounded by C_2 lies in the region between the capacitor plates, so **no current** flows across it.

$$\oint_{C_2} \vec{B} \cdot d\vec{l} = 0 \quad \dots (3)$$

Imagine the loops C_1 and C_2 to be infinitesimally close to each other, as shown. Then we must have

$$\oint_{C_1} \vec{B} \cdot d\vec{l} = \oint_{C_2} \vec{B} \cdot d\vec{l} \quad \dots (4)$$

This result is **inconsistent** with the equations (2) and (3). So a need for modifying Ampere's law was felt by Maxwell.

Maxwell's modification of Ampere's Law:

To modify Ampere's law, Maxwell followed a **symmetry consideration**. By **Faraday's law**, a changing magnetic field induces an electric field, hence a changing electric field must induce a magnetic field. As currents are the usual sources of magnetic fields, a changing electric field must be associated with a current. Maxwell called this current as the **displacement current** to distinguish it from the usual conduction current caused by the drift of electrons.

Displacement current is that current which comes into existence, in addition to the conduction current, whenever the electric field and hence the electric flux changes with time.

To maintain the dimensional consistency, the displacement current is given the form:

$$I_d = \epsilon_0 \frac{d\phi}{dt}$$

Where $\phi = \text{electric field} \times \text{area} = EA$, is the electric flux across the loop.

Therefore, total current across the closed loop = $I_c + I_d = I_c + \epsilon_0 \frac{d\phi}{dt}$

Hence the **modified form of the Ampere's law** is,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left[I + \epsilon_0 \frac{d\phi}{dt} \right]$$

Unlike the conduction current, the displacement current exists whenever the electric field and hence the electric flux is changing with time. Thus according to Maxwell, the source of a magnetic field is not just the conduction electric current due to flowing charges, but also the time varying electric field. Hence the total current I is the sum of the conduction current I_c and the displacement current I_d .

$$I_c + I_d = I_c + \epsilon_0 \frac{d\phi}{dt} \quad \dots (5)$$

Consistency of Ampere's circuital law:

For loop C_1 , there is no electric flux ($\phi=0$). Therefore, from equation (5), we have

$$\oint_{C_1} \vec{B} \cdot d\vec{l} = \mu_0 I \quad \dots (6)$$

For loop C_2 , conduction current $I_c = 0$ but $I_d \neq 0$, because a time varying electric field exists in the region between the capacitor plates. Hence,

$$\oint_{C_2} \vec{B} \cdot d\vec{l} = \mu_0 \cdot \epsilon_0 \frac{d\phi}{dt}$$

If A is the area of the capacitor plates and q be the charge on the plates of the capacitor at any instant t during the charging process, the electric field in the gap will be

$$E = \frac{q}{\epsilon_0 A}$$

$$EA = \frac{q}{\epsilon_0}$$

$$\text{Or flux } \phi = \frac{q}{\epsilon_0}$$

$$\oint_{C_2} \vec{B} \cdot d\vec{l} = \mu_0 \cdot \epsilon_0 \frac{d\phi}{dt} = \mu_0 \cdot \epsilon_0 \frac{d(q/\epsilon_0)}{dt} = \mu_0 \frac{dq}{dt}$$

$$\oint_{C_2} \vec{B} \cdot d\vec{l} = \mu_0 I$$

This agrees with the equation (6), proving the consistency of the **Ampere's modified law** (5).

Property of continuity:

The sum ($I_c + I_d$) has the important property of continuity along any closed path even when individually I_c and I_d may not be continuous. In figure, for example, a current I_c enters one plate and leaves the other plate of the capacitor. The conduction current I_c is not continuous across the capacitor gap as no charge is transported across the gap. The displacement current I_d is zero outside the capacitor plate and in the gap, it has the value

$$\epsilon_0 \frac{d\phi}{dt} = \epsilon_0 \frac{d(q/\epsilon_0)}{dt} = \frac{dq}{dt} \text{ which is exactly the value of the conduction current in the lead wires.}$$

Thus the displacement current satisfies the basic condition that the current is **continuous**. The term $I_c + I_d$ has the same value along the entire path, although **individually** both the **currents are discontinuous**.

Clearly, inside the capacitor we have no conduction current and only displacement current while outside the capacitor we have only conduction current and no displacement current. However, I_c is larger than I_d in a conducting medium while I_d is larger than I_c in an insulating medium.

Consequences of displacement current:

The concept of displacement current has made the laws of electricity and magnetism symmetrical. According to Faraday's law of electromagnetism induction, the magnitude of induced **emf** is equal to the rate of change of **magnetic flux**. But the emf between two points is the work done taking a unit charge from one point to another against the electrostatic forces. This implies the existence of an electric field in the region. So Faraday's law simply states that a time varying magnetic field gives rise to an electric field.

By symmetry, a time varying electric field should give rise to a magnetic field. This is an important consequence of displacement current which is a source of magnetic field.

Another very important consequence of the symmetry of electricity and magnetism is the existence of **electromagnetic waves**, so important for modern communication.

Important properties of displacement current are as follows:

1. Displacement current exists wherever there is a change in electric flux. Unlike conduction current, it does not exist under steady conditions.
2. It is not a current. It only adds to current density in Ampere's circuital law. As it produces magnetic field, so it is called a current.
3. The magnitude of displacement current is equal to the rate of displacement of charge from one capacitor plate to the other.

4. Together with the conduction current, displacement current satisfies the property of continuity.

Maxwell's Equations

Maxwell found that all the basic principles of electromagnetism can be formulated in terms of **four fundamental equations** called the **Maxwell's Equations**. Assuming that no magnetic or dielectric material; is present, the four basic equations can be stated as follows:

1. **Gauss law of Electrostatics** – This law states that the electric flux through a closed surface is $1/\epsilon_0$ (for vacuum) times the total charge **q** enclosed by the surface **S**.

$$\oint \vec{E} \cdot \vec{dS} = \frac{q}{\epsilon_0}$$

Proof:

For the sake of simplicity, we prove Gauss's theorem for an isolated positive point charge **q**. Suppose the surface **S** is a sphere of radius **r** centered on **q**. Then surface **S** is a **Gaussian surface**.

Electric field at any point on **S** is $E = \frac{q}{4\pi\epsilon_0 r^2}$. (For vacuum)

This field points radially outward at all points on **S**. also any area element points radially outwards, so it is parallel to \vec{E} , i.e., $\theta = 0^\circ$.

\therefore Flux through area \vec{dS} is $\vec{d\phi} = \vec{E} \cdot \vec{dS} = E \cdot dS \cdot \cos 0^\circ = EdS$

Total flux through surface **S** is $\phi = \oint_S d\phi = E \oint_S dS = E \times (\text{Total area of sphere})$

$$= \frac{q}{4\pi\epsilon_0 r^2} \times 4\pi r^2 = \frac{q}{\epsilon_0}$$

This proves **Gauss's theorem**.

2. **Gauss law of Magnetism** – By Gauss's law of Electrostatics,

$$\oint \vec{E} \cdot \vec{dS} = \frac{q}{\epsilon_0}$$

Suppose that the closed surface **S** encloses an electric dipole which consists of two equal and opposite charges. Then the total charge enclosed by **S** is zero so that the surface integral of the electrostatic field of a dipole over the closed surface is also zero, i.e.,

$$\oint_S \vec{E}_{\text{dipole}} \cdot \vec{dS} = 0$$

Now a magnetic field is produced only by a magnetic dipole because **isolated magnetic poles do not exist**, so the above equation for a magnetic field can be written as

$$\oint \vec{B} \cdot \vec{dS} = 0$$

According to this law, the magnetic flux, through any closed surface is always zero.

i.e., $\oint \vec{B} \cdot \vec{dS} = 0$

This law implies that isolated magnetic poles or magnetic charges do not exist, i.e., it explains the **absence of magnetic monopoles**.

3. **Faraday's law of electromagnetic induction (Maxwell-Faraday Equation)** – This law tells that a changing magnetic field induces an electric field. According to this law, the induced **emf** set up in a closed circuit is equal to the rate of change of magnetic flux linked with the closed circuit.

$$\oint \vec{E} \cdot d\vec{l} = - \frac{d\phi}{dt}$$

$$\xi \text{ (emf)} = - \frac{d}{dt} [\oint \vec{B} \cdot d\vec{S}]$$

The negative sign here is in accordance with the **Lenz's law**. It states that the direction of the induced current in a circuit is such that it opposes the cause or the change which produces it.

4. **Modified Ampere's Law** – This law states that the surface integral of the magnetic field around any closed circuit is equal to μ_0 times the total current (the sum of conduction and displacement current) threading the closed circuit.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left[I + \epsilon_0 \frac{d\phi}{dt} \right]$$

This law implies that both conduction as well as displacement current **produces a magnetic field**.

Poynting Vector Theorem

Let us consider an electromagnetic wave travelling along the x-axis with the magnetic and electric fields **H** and **E** confined in the transmission planes along z and y-axis respectively. This wave thus propagates along the direction of propagation vector (**E** x **H**). Taking the divergence of Poynting vector in free space, i.e.

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}) &= \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \\ &= - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \\ &= - \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) \\ &= - \left(\epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} \right) \\ &= - \left(\frac{1}{2} \epsilon_0 2\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{2} \mu_0 2\mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} \right) \\ &= - \left(\frac{1}{2} \epsilon_0 \frac{\partial E^2}{\partial t} + \frac{1}{2} \mu_0 \frac{\partial H^2}{\partial t} \right) \\ &= - \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 \right) \end{aligned}$$

Considering that surface **S** bounds a volume **V** and integrating the above relation over the volume **V**, we get

$$\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) = - \frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 \right) dV$$

On applying divergence theorem to the left hand side term of above relation, we get

$$\int_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = - \frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \mu_0 H^2 \right) dV$$

The term on the right hand side within the integral of the above equation represents the sum of energies of electric and magnetic fields. Hence the right hand side of the above equation represents the amount of energy transferred over the volume \mathbf{V} in one second i.e., it represents the rate of flow of energy over the volume \mathbf{V} .

Energy associated with electric field $U_e = \epsilon_0 E^2/2$ and that with the magnetic field $U_m = \mu_0 H^2/2 = \frac{B^2}{2\mu_0}$.

As $U_m = B^2/2\mu_0 = (E/C)^2/2\mu_0 = \frac{1}{2}\epsilon_0 E^2 = U_e$ (as $E/B = C$ and $C = 1/\sqrt{\mu_0\epsilon_0}$)

which shows that instantaneous energy density associated with the magnetic field is equal to the instantaneous energy density associated with electric field i.e. energy is equally shared by the two fields.

The vector \mathbf{P} ($= \mathbf{E} \times \mathbf{H}$) is interpreted as representing the amount of field energy passing through unit area of surface in unit time normally to the direction of flow of energy. This statement is termed as **Poynting's theorem** and the vector \mathbf{P} is called the **Poynting vector**.

Hence the Poynting vector integrated over a closed surface \mathbf{S} i.e. the term $\oint_{\mathbf{S}} (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S}$

Represents the rate of flow of energy over the surface \mathbf{S} enclosing the volume \mathbf{V} . From above we can say Poynting vector represents the flow of energy over a given volume. Vector \mathbf{P} has the dimension of **watt/m²** and direction of flow is perpendicular to \mathbf{E} and \mathbf{H} i.e. in the direction of the vector $\mathbf{E} \times \mathbf{H}$.

Quantum Physics

Classical mechanics successfully explained the motion of the objects which are directly observable or by a microscope. But the same laws of mechanics failed to describe the actual behavior of particles with atomic dimensions e.g. an electron. Many phenomena like the stability of atom, emission of e.m.-wave from an atom, spectrum of hydrogen atom cannot be explained by classical mechanics. For the explanation of such phenomena, **Planck** in 1900 proposed the **Quantum theory**. Many other scientists like Einstein, de-Broglie, Maxwell, etc. also contributed to the same and proposed many theories.

Wave function and its Characteristics

A **wavefront** at any time is defined a continuous locus of all the neighboring particles vibrating in the same phase, i.e., which are in the same state of vibration. A wavefront can be of spherical, plane or cylindrical type depending upon the light source. And Ψ is the **wave displacement** for the de-Broglie waves at any time t . It is also called the **wave function**, the **probability** of finding the particle within a volume.

Characteristics:

1. **The wave function is normalizable** – According to the probability interpretation of Ψ , the probability of finding the particle within volume element $d\tau$ is expressed as $|\Psi^2|d\tau$. Since the particle is certainly somewhere within the volume element, the integral of $|\Psi^2|d\tau$ over the whole space must be unity, i.e.,

$$\int_{-\infty}^{+\infty} |\Psi^2| d\tau = 1$$

A wave function that satisfies this condition is known as normalizable. Every acceptable function must be normalizable. The above condition is known as normalization condition.

2. **The wave function is single-valued** – If Ψ has more than one value at a point, then there will be more than one value of probability of finding the particle at that point. This is obviously ridiculous.
3. **The wave function is finite everywhere** – Let Ψ is infinite at a particular point. It means that there is an infinitely large probability of finding the particle at that point. This violates the uncertainty principle. So, Ψ must have a finite or zero value at that point.
4. **The wave function is continuous and has a continuous first derivative everywhere** – This is necessary from Schrodinger equation itself which shows that $d^2\Psi/dx^2$ must be finite everywhere. This is only possible if $d\Psi/dx$ has no discontinuity at any boundary where the potential changes. Furthermore, the existence of $d\Psi/dx$ as a continuous function implies that Ψ too is continuous across boundary.
5. **Orthogonal and Orthonormal functions** – If the product of a function $\Psi_1(x)$ and complex conjugate $\Psi_2(x)^*$ of a function $\Psi_2(x)$ vanishes when integrated with respect to x over the interval $a \leq x \leq b$, i.e., if

$$\int_a^b \Psi_1(x) \Psi_2(x)^* dx = 0$$

then $\Psi_1(x)$ and $\Psi_2(x)$ are said to be mutually orthogonal or simply orthogonal in the interval (a, b) ,

$$\text{If } \int_a^b \Psi_2(x) \Psi_2(x)^* dx = 1$$

then function is said to be normalized. The functions which are orthogonal and also normalized are called as orthonormal.

A set of functions $\Psi_1(x), \Psi_2(x), \Psi_3(x) \dots \Psi_n(x)$ is orthonormal when

$$\int_a^b \Psi_1(x) \Psi_2(x)^* dx = 0$$

and
$$\int_a^b \Psi_2(x) \Psi_2(x)^* dx = 1.$$

6. **Degenerate Eigen values** – If corresponding to a single eigen value, there is only one eigen function, then it is known as non-degenerate eigen function. If corresponding to a single eigen value, there are a number of eigen functions, then these functions are known as degenerate eigen functions.

Schrodinger's Wave Equations

Schrodinger time-dependent wave equation:

According to the **de-Broglie** theory, a particle of mass **m** is always associated with a wave whose wavelength is given by $\lambda = h/mv$. If the particle has wave properties, it is expected that there should be some sort of equation which describes the behavior of the particle. Consider a system of stationary waves associated with a particle. Let x, y, z be the coordinates of the particle, and Ψ , the wave displacement for the de-Broglie waves at any time **t**. The classical differential equation of a wave motion is given by

$$\frac{\partial^2 \Psi}{\partial t^2} = v^2 \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) = v^2 \nabla^2 \Psi \quad \dots (1)$$

Where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

[∇^2 being the **Laplacian operator** and **v** is the **wave velocity**.]

The solution of equation (1) gives Ψ as a particle displacement in terms of time, i.e.,

$$\Psi(x, y, z, t) = \Psi_0(x, y, z) e^{-i\omega t} \quad \dots (2)$$

where $\Psi_0(x, y, z)$ is a function of x, y, z and gives the amplitude at the point considered. Equation (2) can also be expressed as

$$\Psi(\vec{r}, t) = \Psi_0(\vec{r}) e^{-i\omega t} \quad \dots (3)$$

Differentiating equation (3) twice we get

$$\frac{\partial^2 \Psi}{\partial t^2} = -\omega^2 \Psi_0(\vec{r}) e^{-i\omega t} = -\omega^2 \Psi$$

Substituting the value of $\frac{\partial^2 \Psi}{\partial t^2}$ in equation (1), we have

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} + \frac{\omega^2}{v^2} \Psi = 0 \quad \dots (4)$$

But $\omega = 2\pi\nu = 2\pi \frac{v}{\lambda} \left[\frac{\omega}{v} = \frac{2\pi}{\lambda} \right]$

Substituting the value of ω/v in equation (4), we have

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} + \frac{4\pi^2}{\lambda^2} \Psi = 0$$

$$\nabla^2 \Psi + \frac{4\pi^2}{\lambda^2} \Psi = 0 \quad \dots (5)$$

Now from the de-Broglie relation $\lambda = \frac{h}{mv}$.

$$\nabla^2 \Psi + \frac{4\pi^2}{h^2} m^2 v^2 \Psi = 0 \quad \dots (6)$$

If \mathbf{E} and \mathbf{V} be the total and potential energies of the particle respectively, then its kinetic energy $1/2mv^2$ is given by

$$\frac{1}{2} m v^2 = E - V$$

or $m^2 v^2 = 2m(E - V) \quad \dots (7)$

From equations (6) and (7), we have

$$\nabla^2 \Psi + \frac{4\pi^2}{h^2} \cdot 2m(E - V) \cdot \Psi = 0$$

or $\nabla^2 \Psi + \frac{8\pi^2 m}{h^2} (E - V) \Psi = 0 \quad \dots (8)$

Equation (8) is known as Schrodinger **time independent** wave equation.

Substituting $\hbar = h/2\pi$ in equation (8), the Schrodinger wave equation can be written as

$$\nabla^2 \Psi + \frac{2m}{\hbar^2} (E - V) \Psi = 0 \quad \dots (9)$$

For a free particle $V = 0$, hence the Schrodinger wave equation for a free particle can be expressed as

$$\nabla^2 \Psi + \frac{2mE}{\hbar^2} \Psi = 0 \quad \dots (10)$$

Schrodinger time-independent wave equation:

The Schrodinger time dependent wave equation may be obtained from Schrodinger time independent wave equation by eliminating \mathbf{E} .

Differentiating equation (3) with respect to \mathbf{t} , we get

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= -i\omega \Psi_0(\vec{r}) e^{-i\omega t} = -i(2\pi\nu) \Psi_0(\vec{r}) e^{-i\omega t} \\ &= -2\pi i \nu \Psi = 2\pi i \Psi \frac{E}{h} \quad (\because E = h\nu \text{ or } \nu = E/h) \\ E\Psi &= i\hbar \frac{\partial \Psi}{\partial t} \quad \dots (11) \end{aligned}$$

Substituting the value of $E\Psi$ in Schrodinger time dependent wave equation, we get

$$\begin{aligned} \nabla^2 \Psi + \frac{2m}{\hbar^2} \left[i\hbar \frac{\partial \Psi}{\partial t} - V\Psi \right] &= 0 \\ \nabla^2 \Psi &= -\frac{2m}{\hbar^2} \left[i\hbar \frac{\partial \Psi}{\partial t} - V\Psi \right] \\ -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi &= i\hbar \frac{\partial \Psi}{\partial t} \quad \dots (12) \end{aligned}$$

This equation is known as Schrodinger **time-independent** wave equation.

Equation (12) can be written as

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

or $\mathbf{H\Psi = E\Psi}$... (13)

where $H = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)$ and $E = i\hbar\frac{\partial}{\partial t}$.

Equation (13) describes the motion of a **non-relativistic material particle**.

Particle in a Box

Consider the case of a **single particle** i.e., a gas molecule mass **m**, confined within a rectangular box with edges parallel to X, Y and Z axes as shown. Let the sides of rectangular box be **a, b, c** respectively. The particle can move freely within the region $0 < x < a$, $0 < y < b$ and $0 < z < c$ i.e., inside the box where potential **V** is zero i.e.,

$$V(x,y,z) = 0 \quad < 0 < x < a$$

$$V(x,y,z) = 0 \quad < 0 < y < b$$

And $V(x,y,z) = 0 \quad 0 < z < c$

The potential rises suddenly to have a very large value at the boundaries i.e., the potential outside the box is **infinite**. The **Schrodinger wave equation** inside the box is given by

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} + \frac{2m}{\hbar^2}E\Psi = 0 \quad \dots (1)$$

This is a partial differential equation in three independent variables and may be solved by the method of separation of variables. The solution of equation (1) is of the form

$$\Psi(x,y,z) = X(x).Y(y).Z(z) = XYZ \quad \dots (2)$$

where X(x) is a function of **x** alone, Y(y) is a function of **y** alone and Z(z) is a function of **z** alone.

Substituting the value of **Ψ** from equation (2) in equation (1) and dividing by X(x), Y(y), Z(z), we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + \frac{2mE}{\hbar^2} = 0 \quad \dots (3)$$

The equation may be written as

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - \frac{2mE}{\hbar^2} \quad \dots (4)$$

The left hand side of equation (4) is a function of **x** alone, while the right hand side is a function of **y** and **z** is independent of **x**. Both sides are equal to each other. This is only possible when they are separately equal to a **constant quantity** i.e.,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k_x \quad \dots (5)$$

and $-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - \frac{2mE}{\hbar^2} = k_x$

or $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - \frac{2mE}{\hbar^2} - k_x \quad \dots (6)$

In equation (6), the left hand side is a function of **y** alone while right hand side is a function of **z** and is **independent of y**. If above equation is to be satisfied both sides must be equal to a constant say **k_y** i.e.,

$$\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = k_y \quad \dots (7)$$

and $-\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} - \frac{2mE}{\hbar^2} - k_x = k_y$

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = -\frac{2mE}{\hbar^2} - k_x - k_y \quad \dots (8)$$

Again we have $\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = k_z \quad \dots (9)$

And $-\frac{2mE}{\hbar^2} - k_x - k_y = k_z$

$$-\frac{2mE}{\hbar^2} = k_x + k_y + k_z \quad \dots (10)$$

For convenience, we introduce

$$-\frac{2mE_x}{\hbar^2} = k_x, \quad -\frac{2mE_y}{\hbar^2} = k_y, \quad -\frac{2mE_z}{\hbar^2} = k_z$$

Now the differential equations in x, y and z coordinates may be written as

$$\frac{\partial^2 X}{\partial x^2} + \frac{2mE_x}{\hbar^2} X = 0 \quad \dots (11a)$$

$$\frac{\partial^2 Y}{\partial y^2} + \frac{2mE_y}{\hbar^2} Y = 0 \quad \dots (11b)$$

$$\frac{\partial^2 Z}{\partial z^2} + \frac{2mE_z}{\hbar^2} Z = 0 \quad \dots (11c)$$

The general solution of equation (11a) will be a **sine function** of arbitrary amplitude, frequency and phase, i.e.,

$$X(x) = A \sin(Bx+C) \quad \dots (12)$$

Where **A**, **B** and **C** are constants whose values are determined by boundary conditions.

We know that $|\Psi|^2$, represents the probability of finding the particle at any point within the box. Therefore, $|X(x)|^2$ which is a function of x coordinates only, represents the probability of finding the particle at any point along the x-axis. As the potential is **very high** at the walls of the box, the probability of finding the particle at the walls will be zero, i.e.,

$$|X(x)| = 0 \quad \text{when } x = 0 \text{ and } x = a$$

or $X(x) = 0 \quad \text{when } x = 0 \text{ and } x = a$

Using these boundary conditions in equation (12), we have

$$0 = A \sin(0 + C) \quad A \neq 0$$

$$\therefore \sin C = 0 \quad \text{and} \quad 0 = A \sin(Ba + C)$$

Which yields $\sin(Ba) = 0$ or $Ba = n_x \pi$

or $B = \frac{n_x \pi}{a}$ where n_x is a positive integer.

$$X(x) = A \sin\left(\frac{n_x \pi x}{a}\right) \quad \dots (13)$$

Applying the normalization condition between $x = 0$ to $x = a$, we have

$$\int_0^a |X(x)|^2 dx = 1$$

$$\text{or } \int_0^a \left| A \sin\left(\frac{n_x \pi x}{a}\right) \right|^2 dx = 1$$

$$\text{or } A^2 \int_0^a \left| \sin\left(\frac{n_x \pi x}{a}\right) \right|^2 dx = 1$$

$$\text{or } A^2 \frac{a}{2} = 1 \text{ or } A = \sqrt{\frac{2}{a}}$$

Therefore,

$$X(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi x}{a}\right) \quad \dots (14)$$

Similarly, we can solve equations (11b) and (11c) to obtain

$$Y(y) = \sqrt{\frac{2}{b}} \sin\left(\frac{n_y \pi y}{b}\right) \quad \dots (15)$$

$$\text{and } Z(z) = \sqrt{\frac{2}{c}} \sin\left(\frac{n_z \pi z}{c}\right) \quad \dots (16)$$

The complete wave function

$\Psi_{n_x, n_y, n_z}(x, y, z)$ has the form

$$\begin{aligned} \Psi_{n_x, n_y, n_z}(x, y, z) &= X(x), Y(y), Z(z) \\ &= \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi x}{a}\right) \sqrt{\frac{2}{b}} \sin\left(\frac{n_y \pi y}{b}\right) \sqrt{\frac{2}{c}} \sin\left(\frac{n_z \pi z}{c}\right) \\ &= \frac{2\sqrt{2}}{\sqrt{abc}} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{b}\right) \sin\left(\frac{n_z \pi z}{c}\right) \quad \dots (17) \end{aligned}$$

From equation (14), we have

$$\frac{\partial^2 X}{\partial x^2} = -\left(\frac{n_x \pi}{a}\right)^2 \sqrt{\frac{2}{a}} \sin\frac{n_x \pi x}{a} = -\left(\frac{n_x \pi}{a}\right)^2 X(x) \quad \dots (18)$$

Substituting the value from equation (18) in equation (11a), we have

$$-\left(\frac{n_x \pi}{a}\right)^2 X(x) + \frac{2m}{\hbar^2} E_x X(x) = 0$$

$$\text{or } E_x = \frac{1}{2m} \left(\frac{n_x \pi \hbar}{a}\right)^2 = \frac{n_x^2 \hbar^2}{8ma^2} \quad \dots (19)$$

Similarly,

$$E_y = \frac{n_y^2 \hbar^2}{8mb^2} \quad \dots (20)$$

$$\text{and } E_z = \frac{n_z^2 \hbar^2}{8mc^2} \quad \dots (21)$$

The allowed values of total energy are given by

$$E = E_x + E_y + E_z = \frac{h^2}{8m} \left[\frac{n_x^2}{a^2} + \frac{y^2}{b^2} + \frac{n_z^2}{c^2} \right] \quad \dots (22)$$

Where n_x, n_y, n_z , denote any set of three positive numbers.